Integrals of Polynomial Functions over Spheres and Balls

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1 Definitions and Problem

Let $B_n = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$ be the unit n-dimensional ball and $S_{n-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit $(n-1)$-dimensional sphere. Note that this sphere is the surface of the ball: $S_{n-1} = \partial B_n$.

We want to evaluate the integral

$$I^{B_n}_{a_1 \dots a_n} = \int_{B_n} \left(\prod_{i=1}^{n} x_i^{a_i} \right) \mathrm{d}^n \mathbf{x}$$

of a general monomial term over the n-dimensional ball. It will be helpful to define a similar integral

$$I^{S_{n-1}}_{a_1 \dots a_n} = \int_{S_{n-1}} \left(\prod_{i=1}^{n} x_i^{a_i} \right) \mathrm{d}^{n-1} \mathbf{x}$$

over the surface of the sphere. Finally, an integral over all space but weighted by an n-dimensional unit Gaussian distribution will also be useful:

$$I^{g_n}_{a_1 \dots a_n} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} x_i^{a_i} \right) \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}|\mathbf{x}|^2} \mathrm{d}^n \mathbf{x}.$$

2 Separability of Gaussian Integral

The Gaussian integrand can be written as a product

$$I^{g_n}_{a_1 \dots a_n} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} x_i^{a_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2} \right) \mathrm{d}^n \mathbf{x}.$$

Since each term in the product only depends on x_i, this is a separable integral that is the product of one-dimensional integrals:

$$I^{g_n}_{a_1 \dots a_n} = \prod_{i=1}^{n} \int_{\mathbb{R}} x_i^{a_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2} \, \mathrm{d}x_i = \prod_{i=1}^{n} I^{g_1}_{a_i}.$$

3 Values of One-Dimensional Gaussian Integral $I^{g_1}_a$

The one-dimensional Gaussian integrals can be evaluated by noting

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^a\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right] = (ax^{a-1} - x^{a+1})\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

and therefore by integrating both sides over $\mathbb{R}$,

$$0 = aI^{g_1}_{a-1} - I^{g_1}_{a+1}.$$

The recurrence of values can be started by noting $I^{g_1}_0 = 1$ because the Gaussian is normalised and $I^{g_1}_1 = 0$ because it is the integral of an odd function. Writing the recurrence as $I^{g_1}_{a+2} = (a+1)I^{g_1}_a$ makes it clear that $I^{g_1}_a = 0$ for a odd. For even values,

$$I^{g_1}_2 = 1, \quad I^{g_1}_4 = 3, \quad I^{g_1}_6 = 3\times 5, \quad I^{g_1}_8 = 3\times 5\times 7, \quad \cdots,$$

giving the general formula

$$I^{g_1}_{2a} = \prod_{b=1}^{a}(2b-1) = (2a-1)!! = \frac{(2a)!}{2^a a!}.$$

3.1 Integrals on the Half Real Line $I^{h_1}_a$

The following calculations will also need 1D Gaussian integrals on the half real line defined by

$$I^{h_1}_a = \int_0^\infty x^a\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,\mathrm{d}x.$$

Starting as before but integrating over $[0,\infty)$ gives

$$\left[x^a\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right]_{x=0}^{\infty} = aI^{h_1}_{a-1} - I^{h_1}_{a+1}.$$

The left hand side is equal to $\frac{-1}{\sqrt{2\pi}}$ when $a=0$ and zero otherwise. The recurrence can still be rewritten $I^{h_1}_{a+2} = (a+1)I^{h_1}_a$ valid for $a \geq 0$. The original $a=0$ case gives

$$\frac{-1}{\sqrt{2\pi}} = -I^{h_1}_1 \qquad \Rightarrow \qquad I^{h_1}_1 = \frac{1}{\sqrt{2\pi}}.$$

For even functions we have $I^{h_1}_{2a} = \frac{1}{2}I^{g_1}_{2a}$ and in particular $I^{h_1}_0 = \frac{1}{2}$. The recurrence gives the general formulae

$$I^{h_1}_{2a} = \frac{1}{2}(2a-1)!! \qquad \text{and} \qquad I^{h_1}_{2a+1} = \frac{1}{\sqrt{2\pi}}\prod_{b=1}^{a}2b = \frac{1}{\sqrt{2\pi}}(2a)!! = \frac{1}{\sqrt{2\pi}}2^a a!.$$

4 Relation of Gaussian to Spherical Integral

The Gaussian integrand can also be split into parts that depend on radius $r = |\mathbf{x}|$ and parts that do not:

$$I^{g_n}_{a_1\ldots a_n} = \int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi}^n}e^{-\frac{1}{2}r^2}r^{\sum a_i}\left(\prod_{i=1}^{n}\left(\frac{x_i}{r}\right)^{a_i}\right)\mathrm{d}^n\mathbf{x}.$$

Defining $\mathbf{x} = r\mathbf{u}$ so that $\mathbf{u} \in S_{n-1}$ gives $\mathrm{d}^n\mathbf{x} = r^{n-1}\,\mathrm{d}r\,\mathrm{d}^{n-1}\mathbf{u}$ and

$$\begin{aligned} I^{g_n}_{a_1\ldots a_n} &= \int_0^\infty \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}r^2} r^{\sum a_i} r^{n-1}\,\mathrm{d}r \int_{S_{n-1}} \left(\prod_{i=1}^n u_i^{a_i}\right) \mathrm{d}^{n-1}\mathbf{u} \\ &= \frac{1}{\sqrt{2\pi}^{n-1}} I^{h_1}_{n-1+\sum a_i} I^{S_{n-1}}_{a_1\ldots a_n}. \end{aligned}$$

5 Evaluating the Spherical Integral

By the formula in the previous section,

$$I^{S_{n-1}}_{a_1\ldots a_n} = \frac{\sqrt{2\pi}^{n-1} I^{g_n}_{a_1\ldots a_n}}{I^{h_1}_{n-1+\sum a_i}}.$$

The separability of the Gaussian integrals allows this to be written with only $I^{g_1}_a$ and $I^{h_1}_a$ terms:

$$I^{S_{n-1}}_{a_1\ldots a_n} = \sqrt{2\pi}^{n-1} \frac{\prod_{i=1}^n I^{g_1}_{a_i}}{I^{h_1}_{n-1+\sum a_i}}.$$

6 Evaluating the Ball Integral

Integrating over a smaller sphere of radius r will introduce a factor of r^{n-1} from the change in surface area and a factor of $r^{\sum a_i}$ from the change in the monomial term itself. Thus,

$$I^{B_n}_{a_1\ldots a_n} = \int_0^1 r^{n-1+\sum a_i} I^{S_{n-1}}_{a_1\ldots a_n}\,\mathrm{d}r = \frac{1}{n+\sum a_i} I^{S_{n-1}}_{a_1\ldots a_n}.$$

Using the formula from the previous section, this can be written in terms of $I^{g_1}_a$ and $I^{h_1}_a$:

$$I^{B_n}_{a_1\ldots a_n} = \frac{\sqrt{2\pi}^{n-1}}{n+\sum a_i} \frac{\prod_{i=1}^n I^{g_1}_{a_i}}{I^{h_1}_{n-1+\sum a_i}}.$$